# Chapter 10

# Lie Algebras/Angular momentum

This chapter sits between an introduction to Lie Algebra and a recap of angular momentum and the addition of angular momentum. For those of you who are already comfortable with all things angular momentum this chapter is intended to expose you to the deep mathematics underlying these topics to provide you with a fresh perspective and equip you for future forays into the world of Lie Algebras. That said, for those of you who are fed up of group theory by this point and/or are perhaps a little rusty when it comes to the addition of angular momentum you should be able to largely ignore the group theory and reason your way through these topics based on physical intuition.

There are two main topics we will cover. Firstly, we will introduce the Lie algebras by looking at the example of rotations. In doing so, we will rediscover a lot of what you already know about angular momentum (but frame it slightly different language). Secondly, we will discuss the addition of angular momentum and the relationship between this and the irreducible representations of tensor product representations of rotation groups. In both parts I will closely follow Group Theory in a Nutshell for Physicists (GTNFP). I'm going to copy the most relevant sections here for your convenience (making only minor tweaks) but you might prefer to go and read directly from there.

Also, Joachim's abridged notes (at the top of the moodle) are great for those of you who find this all a bit waffling. He presents it much more concisely than my trimmed (but not so trim) notes based around GTNFP.

## 10.1 Intro to Lie Algebras via Rotations. An trimmed copy of I.3 in GTNFP

### 10.1.1 A little bit at a time

The Norwegian physicist Marius Sophus Lie (1842–1899) had the almost childishly obvious but brilliant idea that to rotate through, say, 29°, you could just as well rotate through a zillionth of a degree and repeat the process 29 zillion times. To study rotations, it suffices to study rotation through infinitesimal angles. Shades of Newton and Leibniz! A rotation through a finite angle can always be obtained by performing infinitesimal rotations repeatedly. As is typical with many profound statements in physics and mathematics, Lie's idea is astonishingly simple. Replace the proverb "Never put off until tomorrow what you have to do today" by "Do what you have to do a little bit at a time."

A rotation through an infinitesimal angle $\theta$ is almost the identity $I$, that is, no rotation at all, and so can be written as

$$R(\theta) \simeq I + A \tag{10.1}$$

Here $A$ denotes some infinitesimal matrix of order $\theta$. The neglected terms in (10.1) are of order $\theta^2$ and smaller.

Let us imagine Lie saying to himself, "Pretend that I slept through trigonometry class and I don't know anything about how rotation matrices look. Instead, I will define rotations as the set of linear transformations on 2-component objects $\boldsymbol{u}' = R\boldsymbol{u}$ and $\boldsymbol{v}' = R\boldsymbol{v}$ that leave $\boldsymbol{u}^T \cdot \boldsymbol{v}$ invariant. I will impose

$$R^T R = I \tag{10.2}$$

and derive (10.1). But according to my brilliant idea, it suffices to solve this condition for rotations infinitesimally close to the identity."

Following Lie, we plug $R \simeq I + A$ into (10.2). Since by assumption $A^2$, being of order $\theta^2$, can be neglected relative to $A$, we have

$$R^T R \simeq (I + A^T)(I + A) \simeq (I + A^T + A) = I \tag{10.3}$$

Thus, this requires $A^T = -A$, namely, that $A$ must be antisymmetric.

But there is basically only one $2 \times 2$ antisymmetric matrix:

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10.4}$$

In other words, the solution of $A^T = -A$ is $A = \theta\mathcal{J}$ for some real number $\theta$. Thus, rotations close to the identity have the form

$$R = I + \theta\mathcal{J} + O(\theta^2) = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} + O(\theta^2) \tag{10.5}$$

The antisymmetric matrix $\mathcal{J}$ is known as the generator of the rotation group. We obtain, without knowing any trigonometry, that under an infinitesimal rotation, $x \to x' \simeq x + \theta y$, and $y \to y' \simeq -\theta x + y$, which is of course consistent with (10.5). We could also obtain this result by drawing an elementary geometrical figure involving infinitesimal angles.

Now recall the identity $e^x = \lim_{N\to\infty}(1 + \frac{x}{N})^N$ (which you can easily prove by differentiating both sides). Then, for a finite (that is, not infinitesimal) angle $\theta$, we have

$$R(\theta) = \lim_{N\to\infty} \left(R\left(\frac{\theta}{N}\right)\right)^N = \lim_{N\to\infty} \left(1 + \frac{\theta\mathcal{J}}{N}\right)^N = e^{\theta\mathcal{J}} \tag{10.6}$$

The first equality represents Lie's profound idea: we cut up the given noninfinitesimal angle $\theta$ into $N$ pieces so that $\theta/N$ is infinitesimal for $N$ large enough and perform the infinitesimal rotation $N$ times. The second equality is just (10.5). For the last equality, we use the identity just mentioned, which amounts to the definition of the exponential.

As an alternative but of course equivalent path to our result, simply assert that we have every right, to leading order, to write $R\left(\frac{\theta}{N}\right) = 1 + \frac{\theta\mathcal{J}}{N} \simeq e^{\frac{\theta\mathcal{J}}{N}}$. Thus

$$R(\theta) = \lim_{N\to\infty} \left(R\left(\frac{\theta}{N}\right)\right)^N = \lim_{N\to\infty} \left(e^{\frac{\theta\mathcal{J}}{N}}\right)^N = e^{\theta\mathcal{J}} \tag{10.7}$$

In calculus, we learned about the Taylor or power series. Taylor said that if we gave him all the derivatives of a function $f(x)$ at $x = 0$ (say), he could construct the function. In contrast, Lie said that, thanks to the multiplicative group structure, he only needs the first derivative of the group element $R(\theta)$ near the identity. Indeed, we recognize that $\mathcal{J}$ is just $\frac{dR(\theta)}{d\theta}\Big|_{\theta=0}$. The reason that Lie needs so much less is of course that the group structure is highly restrictive.

Finally, we can check that the formula $R(\theta) = e^{\theta\mathcal{J}}$ reproduces (10.5) for any value of $\theta$. We simply note that $\mathcal{J}^2 = -I$ and separate the exponential series, using Taylor's idea, into even and odd powers of $\mathcal{J}$:

$$e^{\theta\mathcal{J}} = \sum_{n=0}^{\infty} \frac{\theta^n \mathcal{J}^n}{n!} = \left(\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}\right) I + \left(\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!}\right) \mathcal{J} \tag{10.8}$$

which simplifies to

$$e^{\theta\mathcal{J}} = \cos\theta\, I + \sin\theta\, \mathcal{J} = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{10.9}$$

which is the familiar rotation matrix in 2D that you probably derived for yourself back in high school using trigonometry. Note that this works, because $\mathcal{J}$ plays the same role as $i$ in Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$.

### 10.1.2 Lie in higher dimensions

The power of Lie now shines through when we want to work out rotations in higher-dimensional spaces. All we have to do is satisfy the two conditions $R^T R = I$ and $\det R = 1$. Lie shows us that the first condition, $R^T R = I$, is solved immediately by writing $R \simeq I + A$ and requiring $A = -A^T$, namely, that $A$ be antisymmetric. That's it. We could be in a zillion-dimensional space, but still, the rotation group is fixed by requiring $A$ to be antisymmetric.

But it is very easy to write down all possible antisymmetric $N$-by-$N$ matrices! For $N = 2$, there is only one, namely, the $\mathcal{J}$ introduced earlier. For $N = 3$, there are basically three of them:

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10.10}$$

Any 3-by-3 antisymmetric matrix can be written as $A = \theta_x \mathcal{J}_x + \theta_y \mathcal{J}_y + \theta_z \mathcal{J}_z$, with three real numbers $\theta_x$, $\theta_y$, and $\theta_z$. The three 3-by-3 antisymmetric matrices $\mathcal{J}_x$, $\mathcal{J}_y$, $\mathcal{J}_z$ are known as generators. They generate rotations, but are of course not to be confused with rotations, which are by definition 3-by-3 orthogonal matrices with determinant equal to 1.

One upshot of this whole discussion is that any 3-dimensional rotation (not necessarily infinitesimal) can be written as

$$R(\theta) = e^{\theta_x \mathcal{J}_x + \theta_y \mathcal{J}_y + \theta_z \mathcal{J}_z} = e^{\Sigma_i \theta_i \mathcal{J}_i} \tag{10.11}$$

(with $i = x, y, z$) and is thus characterized by three real numbers $\theta_x$, $\theta_y$, and $\theta_z$. As I said, those readers who have suffered through the rotation of a rigid body in a course on mechanics surely would appreciate the simplicity of studying the generators of infinitesimal rotations and then simply exponentiating them.

To mathematicians, physicists often appear to use weird notations. There is not an $i$ in sight, yet physicists are going to stick one in now. If you have studied quantum mechanics, you know that the generators $\mathcal{J}$ of rotation studied here are related to angular momentum operators. You would also know that in quantum mechanics observables are represented by hermitean operators or matrices. In contrast, in our discussion, the $\mathcal{J}$s come out naturally as real antisymmetric matrices and are thus antihermitean. To make them hermitean, we multiply them by some multiples of the imaginary unit $i$. Thus, define

$$J_x \equiv -i\mathcal{J}_x, \quad J_y \equiv -i\mathcal{J}_y, \quad J_z \equiv -i\mathcal{J}_z \tag{10.12}$$

and write a general rotation as

$$R(\theta) = e^{i\Sigma_i \theta_i J_i} = e^{i\boldsymbol{\theta}\cdot\boldsymbol{J}} \tag{10.13}$$

Treating the three real numbers $\theta_j$ and the three matrices $J_j$ as two 3-dimensional vectors.

*Exercise:* Write down the generators of rotations in 4-dimensional space. At least count how many there are.

### 10.1.3 Structure constants

In general, rotations do not commute. Following Lie, we could try to capture this essence of group multiplication by focusing on infinitesimal rotations.

Let $R \simeq I + A$ be an infinitesimal rotation. For an arbitrary rotation $R'$, consider

$$RR'R^{-1} \simeq (I + A)R'(I - A) \simeq R' + AR' - R'A \tag{10.14}$$

(where we have consistently ignored terms of order $A^2$). If rotations commute, then $RR'R^{-1}$ would be equal to $R'$. Thus, the extent to which this is not equal to $R'$ measures the lack of commutativity. Now, suppose $R'$ is also an infinitesimal rotation $R' \simeq I + B$. Then

$$RR'R^{-1} \simeq I + B + AB - BA \tag{10.15}$$

which differs from $R' \simeq I + B$ by the matrix

$$[A, B] \equiv AB - BA, \tag{10.16}$$

known as the commutator between $A$ and $B$.

For $SO(3)$, for example, $A$ is a linear combination of the $J$s, which we shall call the generators of the Lie algebra of $SO(3)$. Thus, we can write

$$A = i\Sigma_i \theta_i J_i \quad \text{and similarly} \quad B = i\Sigma_j \theta'_j J_j. \tag{10.17}$$

Hence

$$[A, B] = i^2 \Sigma_{ij} \theta_i \theta'_j [J_i, J_j], \tag{10.18}$$

and so it suffices to calculate the commutators $[J_i, J_j]$ once and for all.

Lie's great insight is that the preceding discussion holds for any group whose elements $g(\theta_1, \theta_2, \cdots)$ are labeled by a set of continuous parameters such that $g(0, 0, \cdots)$ is the identity $I$. (For example, the continuous parameters would be the angles $\theta_i$, $i = 1, 2, 3$ in the case of $SO(3)$.) For these groups, now known as Lie groups, this is what you do in four easy steps:

1. Expand the group elements around the identity by letting the continuous parameters go to zero: $g \simeq I + A$.

2. Write $A = i\sum_a \theta_a T_a$ as a linear combination of the generators $T_a$ as determined by the nature of the group.
3. Pick two group elements near the identity: $g_1 \simeq I + A$ and $g_2 \simeq I + B$. Then

$$g_1 g_2 g_1^{-1} \simeq I + B + [A, I + B] \simeq I + B + [A, B].$$

The commutator $[A, B]$ captures the essence of the group near the identity.
4. As in step 2, we can write $B = i\sum_b \theta'_b T_b$ as a linear combination of the generators $T_b$. Similarly, we can write $[A, B]$ as a linear combination of the generators $T_c$. (We know this because, for $g_1$ and $g_2$ near the identity, $g_1 g_2 g_1^{-1}$ is also near the identity.) Plugging in, we then arrive at the analog of (10.18) for any continuous group, namely, the commutation relations

$$[T_a, T_b] = i f_{abc} T_c \tag{10.19}$$

The commutator between any two generators can be written as a linear combination of the generators.

The commutation relations between the generators define a *Lie algebra*, with $f_{abc}$ referred to as the *structure constants* of the algebra.

For example, for $SO(3)$ we have that

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y. \tag{10.20}$$

Therefore, we have that $f_{abc} = \epsilon_{abc}$ where $\epsilon_{abc}$ is the Levi-Civita symbol. These coefficients, i.e., the statement that $f_{abc} = \epsilon_{abc}$ , can thus be used to identify the algebra of $SO(3)$.

Before we move on, let's just take a step back for a second and summarise the jargon we've introduced implicitly introduced in the previous sections.

- A *Lie algebra* $g$ is a linear space spanned by linear combinations $\sum_i \theta_i \mathcal{J}_i$ of the generators of the associated *Lie group* $G$.
- In particular, as Lie groups are differentiable, it is always possible to write an element $g$ of a Lie group $G$ as the exponential of an element $J$ of the corresponding Lie Algebra g. That is,

$$g = \{J | e^{iJ} \in G\}. \tag{10.21}$$

- The commutation relations of the generators $J_j$ (i.e., a basis for g) are the *structure constants* of the group and can be used to identify the Lie Algebra g (and thereby the corresponding Lie group $G$).

In practise, similarly to the case with groups, as physicists we are often more comfortable working with representations of the generators (i.e, a basis) of the Lie Algebra (or just *a representation the Lie algebra*) than with the Lie Algebra itself.

*So, what does it mean to represent the Lie algebra?* It means that we are to find matrices such that the commutation relations that define an algebra are satisfied.

In fact, for SO(2) and SO(3) in the proceeding section, we already wrote down representations of these algebras, namely a 2D representation of SO(2) and then a 3D representation of SO(3), *before* identifying the structure constants that more abstractly identify the algebra. For example, in the case of SO(3), three matrices $J_x, J_y$, and $J_z$ such that the commutation relations are satisfied are specified in (10.20).

But note, as with groups, there are multiple possible representations possible for an algebra. We will explore some alternative higher dimensional representations of SO(3) in Section 10.2.

A word of clarification. Strictly speaking, we should distinguish the matrices representing the abstract operators $J_x, J_y$, and $J_z$ from the abstract operators (that satisfy $f_{abc} = \epsilon_{abc}$) themselves. But it would only clutter up things if we introduce more notation. Instead, we follow the physicist's sloppy practice of using $J_x, J_y$, and $J_z$ also to denote the matrices representing the abstract operators $J_x, J_y$, and $J_z$. Similarly, the Lie Algebra and Lie Group as often represented using lower case mathfrak (e.g., so(3)) and upper case (e.g., $SO(3)$) letters respectively. But again, when it is unambiguous we'll just use upper case letters for both cases.

**A note on the relation between representations of Lie groups and Lie algebras...** You may well have noticed that as rotations are given by exponentials of linear combinations of the $J$s, exponentiating the representations of the SO(3) algebra lead to matrices representing the SO(3) rotation group... or, turning this around, if we take a single parameter subgroup of SO(3), e.g., $e^{-iJ_x}$ and look at its derivative at $\theta = 0$ we will get a representation of one of the basis elements of the algebra, e.g.,

$$\frac{d}{d\theta} e^{-iJ_x}\Big|_{\theta=0} = -iJ_x \,. \tag{10.22}$$

Or even more explicitly any rotation in 3D can be decomposed into rotations around the $x$, $y$ and $z$ axes respectively:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10.23}$$

and we can quickly check that differentiating each of these does give back the generators calculated earlier. For example,

$$\frac{d}{d\theta} R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = -iJ_x \,. \tag{10.24}$$

In fact, you can always get a representation of an algebra in this manner from a representation of the group.

Concretely, let $G$ be a matrix Lie group with Lie algebra g. If $R$ is a representation of $G$ on $V$, then there exists a unique representation $r$ of g on $V$ given by

$$r(X) = \frac{d}{d\theta}\left(R(e^{\theta X})\right)\Big|_{\theta=0}, \quad \text{for all } X \in g.$$

We call $r$ the representation of g induced by $R$.

However, the converse is not always true. You do not always get a representation of the group on exponentiation. You do for most groups - namely simply connected groups. But there are representations of non-simply connected groups where this is not strictly true. We're not going to dive down this conceptually fiddly rabbit hole in this course.

**This is non-examinable.** For completeness we note that a mathematician might define a Lie algebra more abstractly as a vector space g over a field $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ (for us usually over $\mathbb{C}$) with a *Lie bracket* $[\cdot,\cdot] : g \times g \to g$, which satisfies the following axioms holding for all $X_1, X_2, X_3 \in g$ and $a, b \in \mathbb{F}$,

1. *Antisymmetry*: $[X_1, X_2] = -[X_2, X_1]$.
2. *Bilinearity*: $[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3]$.
3. *Jacobi Identity*: $[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0$.

The standard commutator $[A, B] = AB - BA$ satisfies this properties and is generally the only Lie bracket that will matter in most quantum settings.

### 10.1.4 Rotations in Higher Dimensions

With your experience with (10.10) and (10.6), it is now a cinch for you to generalize and write down a complete set of antisymmetric $N$-by-$N$ matrices.

Start with an $N$-by-$N$ matrix with 0 everywhere. Stick a 1 into the $m$-th row and $n$-th column; due to antisymmetry, you are obliged to put a $-1$ into the $n$-th row and $m$-th column. Call this antisymmetric matrix $\mathcal{J}_{(mn)}$. We put the subscripts $(mn)$ in parentheses to emphasize that $(mn)$ labels the matrix. They are not indices to tell us which element of the matrix we are talking about. As explained before, physicists like Hermite a lot and throw in a $-i$ to define the hermitean matrices $J_{(mn)} = -i\mathcal{J}_{(mn)}$. Explicitly,

$$(J_{(mn)})^{ij} = -i(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}) . \tag{10.25}$$

To repeat, in the symbol $(J_{(mn)})^{ij}$, which we will often write as $J^{ij}_{(mn)}$ for short, the indices $i$ and $j$ indicate, respectively, the row and column of the entry $(J_{(mn)})^{ij}$ of the matrix $J_{(mn)}$, while the indices $m$ and $n$, which I put in parentheses for pedagogical clarity, indicate which matrix we are talking about. The first index $m$ on $J_{(mn)}$ can take on $N$ values, and then the second index $n$ can take on only $(N-1)$ values, since, evidently, $J_{(mn)} = 0$. Also, since $J_{(nm)} = -J_{(mn)}$, we require $m > n$ to avoid double counting. Thus, there are only $\frac{1}{2}N(N-1)$ real antisymmetric $N$-by-$N$ matrices $J_{(mn)}$. The Kronecker deltas in (10.25) merely say what we said in words in the preceding paragraph.

As before, an infinitesimal rotation is given by $R \simeq I + A$ with the most general $A$ a linear combination of the $J_{(mn)}$s: $A = i\sum_{m,n}\theta_{(mn)}J_{(mn)}$, where the antisymmetric coefficients $\theta_{(mn)} = -\theta_{(nm)}$ denote $\frac{1}{2}N(N-1)$ generalized angles. (As a check, for $N = 2$ and 3, $\frac{1}{2}N(N-1)$ equals 1 and 3, respectively.) The matrices $J_{(mn)}$ are the generators of the group $SO(N)$.

Our next task is to work out the Lie algebra for $SO(N)$, namely, the commutators between the $J_{(mn)}$s. You could simply plug in (10.25) and chug away. But a more elegant approach is to work out $SO(4)$ as an inspiration for the general case. First, $[J_{(12)}, J_{(34)}] = 0$, as you might expect, since rotations in the (1-2) plane and in the (3-4) plane are like gangsters operating on different turfs. Next, we tackle $[J_{(23)}, J_{(31)}]$. Notice that the action takes place entirely in the $SO(3)$ subgroup of $SO(4)$, and so we already know the answer: $[J_{(23)}, J_{(31)}] = [J_x, J_y] = iJ_z = iJ_{(12)}$. These two examples, together with antisymmetry $J_{(mn)} = -J_{(nm)}$, in fact take care of all possible cases. In the commutator $[J_{(mn)}, J_{(pq)}]$, there are three possibilities for the index sets $(mn)$ and $(pq)$: (i) they have no integer in common, (ii) they have one integer in common, or (iii) they have two integers in common. The commutator vanishes in cases (i) and (iii), for trivial (but different) reasons. In case (ii), suppose $m = p$ with no loss of generality, then the commutator is equal to $iJ_{(nq)}$.

We obtain, for any $N$,

$$[J_{(mn)}, J_{(pq)}] = i(\delta_{mp}J_{(nq)} + \delta_{nq}J_{(mp)} - \delta_{np}J_{(mq)} - \delta_{mq}J_{(np)}) \tag{10.26}$$

This may look rather involved to the uninitiated, but in fact it simply states in mathematical symbols the last three sentences of the preceding paragraph. First, on the right-hand side, a linear combination of the $J$s (as required by the general argument above) is completely fixed by the first term by noting that the left-hand side is antisymmetric under three separate interchanges: $m \leftrightarrow n$, $p \leftrightarrow q$, and $(mn) \leftrightarrow (pq)$. Next, all those Kronecker deltas just say that if the two sets $(mn)$ and $(pq)$ have no integer in common, then the commutator vanishes. If they do have an integer in common, simply "cross off" that integer. For example, $[J_{(12)}, J_{(14)}] = iJ_{(24)}$ and $[J_{(23)}, J_{(31)}] = -iJ_{(21)} = iJ_{(12)}$.

## 10.2 Lie Algebra of so(3) and Ladder Operators: Creation and Annihilation

*In this section we will consider higher dimensional representations of so(3) and then look into how to find its irreducible representations. This should, similarly to the previous section, feel very familiar. You were essentially already shown how to do this when you were first introduced to quantum angular momentum! However, walking through this carefully will give us the tools we need in the next section to tackle the irreducible representations of tensor product reps of so(3) more carefully (i.e., re-study the additional of angular momentum).*

### 10.2.1 Core idea: how to construct arbitrary dimension irreps of so(3)

Let $j \in \{0, \frac{1}{2}, 1, \dots\}$ be arbitrary. We can construct a representation of so(3), labeled by $j$, given by some eigenvectors $|jm\rangle$ of $J_z$. To do so, we define the following ladder operators:

$$J_- = J_x - iJ_y, \qquad J_+ = J_x + iJ_y.$$

The following properties holds:

1. $J_z|j,m\rangle = m|j,m\rangle$.
2. $[J_z, J_\pm] = \pm J_\pm$.
3. $[J_+, J_-] = 2J_z$.

4. $J_+|j,m\rangle = \sqrt{j(j+1) - m(m+1)}\,|j,m+1\rangle$.
5. $J_-|j,m\rangle = \sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle$.

The first property is a matter of definition. The commutators in 2) and 3) stem from the commutation relations between $J_z, J_x, J_y$ derived in the previous lecture and the definition of $J_+$ and $J_-$ above. The proof for 5) and 6) is a little longer but you should have seen it last year and if you've forgotten the extract from

These properties allow us to compute $J_z$ in the basis $|j,m\rangle$ via:

$$\langle j,m'|J_z|j,m\rangle = m\delta_{m,m'}.$$

We can similarly compute $J_x$ and $J_z$ in this basis.

$$J_x = \frac{1}{2}(J_+ + J_-), \qquad J_y = \frac{1}{2i}(J_+ - J_-).$$

These $J_x, J_y, J_z$ form a basis for a representation of so(3).

**Example for j =1/2.** For example, the $j = \frac{1}{2}$ irrep is, in the $\{|\frac{1}{2},\frac{1}{2}\rangle, |\frac{1}{2},-\frac{1}{2}\rangle\}$ basis :

$$J_x = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad J_y = \frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad J_z = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{10.27}$$

To see this note that is trivially follows from $J_z|j,m\rangle = m|j,m\rangle$ that

$$J_z = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Then, using

$$J_\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j,m\pm 1\rangle,$$

one finds $J_-|1/2,1/2\rangle = |1/2,-1/2\pm 1\rangle$ and $J_+|1/2,-1/2\rangle = |1/2,1/2\pm 1\rangle$

$$J_+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad J_- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Hence,

$$J_x = \frac{1}{2}(J_+ + J_-), \qquad J_y = \frac{1}{2i}(J_+ - J_-),$$

giving

$$J_x = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad J_y = \frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad J_z = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Thus $J_k = \frac{1}{2}\sigma_k$, the Pauli matrices in the $\{|0\rangle, |1\rangle\}$ basis.

**Example for j =1.** Similarly, it is straightforward (if a little messier) to compute the $j = 1$ irrep in the $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ basis. You get:

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad J_z = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

These correspond to the spin-1 irreducible representation of so(3)

You might be wondering at this point - didn't we see another 3D representation of so(3) back in Eq. ???? These ones in case you've forgotten and are too lazy to flick back:

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are the standard generators of infinitesimal rotations in $\mathbb{R}^3$. They satisfy the commutation relations

$$[J_i, J_j] = \epsilon_{ijk} J_k,$$

and form the defining (vector) representation of so(3). These two realisations describe the same Lie algebra and are related by a change of basis: the first acts on Cartesian vectors in $\mathbb{R}^3$, while the second acts on states in the spherical (angular momentum) basis. That is, they are *equivalent* representations.

**Irreducible representations of Lie algebras** You'll notice that I'm calling these representations we have just found 'irreps'. But I've not yet defined what we mean by an 'irreducible' representation of a Lie Algebra. Intuitively, the notion is directly analogous to that for group representations.

**Definition 10.2.1** (Reducible and irreducible representations of a Lie algebra (via generators - informal))**.** Let g be a Lie algebra and let

$$\{X_1, \dots, X_k\}$$

be a set of generators of g.

- The representation is completely-reducible if and only if there exists a basis of $V$ in which all generator matrices $X_i$ can be written simultaneously in block-diagonal form.
- In all algebras we will care about in this course[1] if a representation is completely-reducible it is also reducible in the sense of having no invariant subspaces.
- If a representation is not reducible it is called irreducible.
- Therefore, for our purposes, if the generators of the algebra are not simultaneously diagonalisable the representation is irreducible.

In the case of the Lie algebra so(3), it suffices to consider the generators $J_x, J_y, J_z$. If there exists a basis in which these three matrices are simultaneously block-diagonal, then the representation is reducible; otherwise, it is irreducible. For example, clearly there is no single basis in which *all* the Pauli matrices are block-diagonal and so Eq. (10.27) is irreducible as claimed.

Earlier in the course we considered finite groups, for which the number of inequivalent irreducible representations is necessarily finite. In contrast, when dealing with a continuous (Lie) group such as $SO(3)$, the situation is fundamentally different. Owing to the continuous nature of the group, one finds an infinite family of inequivalent irreducible representations, labeled in this case by the spin quantum number. Eg. for so(3) these are labeled by

$$j \in \left\{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots\right\},$$

[1] e.g., all semisimple Lie algebras including so(n) and su(n). In handwavey terms an algebra is semisimple is it built from simple pieces, no "translation-like" directions. More technically, if it has no non-zero abelian ideals.

and the corresponding representation has dimension $2j + 1$. This abundance of irreducible representations is therefore a natural and expected feature of continuous symmetry groups.

### 10.2.2 A recap of ladder operators and a derivation of properties 5. and 6. above. Can skip if you already know this.

*(This is a trimmed copy of (IV.2 of GTNFP))*

Since the three generators $J_x, J_y$, and $J_z$ do not commute, they cannot be simultaneously diagonalized, as explained in the review of linear algebra. But we can diagonalize one of them. Choose $J_z$, and work in a basis in which $J_z$ is diagonal.

The move that breaks the problem wide open should be very familiar to you: it is akin to going from the 2-dimensional coordinates $x, y$ to the complex variable $z = x + iy, z^* = x - iy$, and from a transversely polarized electromagnetic wave to a circularly polarized electromagnetic wave. Define $J_\pm \equiv J_x \pm iJ_y$. Then we can rewrite (10.20) as

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z. \tag{10.28}$$

Write the eigenvector of $J_z$ with eigenvalue $m$ as $|m\rangle$; in other words,

$$J_z|m\rangle = m|m\rangle. \tag{10.29}$$

Since $J_z$ is hermitean, $m$ is a real number. What we are doing is going to a basis in which $J_z$ is diagonal; according to (10.28), $J_\pm$ cannot be diagonal in this basis. Now consider the state $J_+|m\rangle$ and act on it with $J_z$:

$$J_zJ_+|m\rangle = (J_+J_z + [J_z, J_+])|m\rangle = (J_+J_z + J_+)|m\rangle = (m + 1)J_+|m\rangle, \tag{10.30}$$

where the second equality follows from (10.28). (Henceforth, we will be using (10.28) repeatedly without bothering to refer to it.)

Thus, $J_+|m\rangle$ is an eigenvector (or eigenstate; these terms are used interchangeably) of $J_z$ with eigenvalue $m + 1$. Hence, by the definition of $|m\rangle$, the state $J_+|m\rangle$ must be equal to the state $|m + 1\rangle$ multiplied by some normalization constant; in other words, we have

$$J_+|m\rangle = c_{m+1}|m + 1\rangle, \tag{10.31}$$

with the complex number $c_{m+1}$ to be determined. Similarly,

$$J_zJ_-|m\rangle = (J_-J_z + [J_z, J_-])|m\rangle = (J_-J_z - J_-)|m\rangle = (m - 1)J_-|m\rangle, \tag{10.32}$$

from which we conclude that

$$J_-|m\rangle = b_{m-1}|m - 1\rangle, \tag{10.33}$$

with some other unknown normalization constant.

It is very helpful to think of the states $\cdots, |m - 1\rangle, |m\rangle, |m + 1\rangle, \cdots$ as corresponding to rungs on a ladder. The result $J_+|m\rangle = c_{m+1}|m + 1\rangle$ tells us that we can think of $J_+$ as a "raising operator" that enables us to climb up one rung on the ladder, going from $|m\rangle$ to $|m + 1\rangle$. Similarly, the result $J_-|m\rangle = b_{m-1}|m - 1\rangle$ tells us to think of $J_-$ as a "lowering operator" that enables us to climb down one rung on the ladder. Collectively, $J_\pm$ are referred to as ladder operators.

To relate $b_m$ to $c_m$, we invoke the hermiticity of $J_x, J_y$, and $J_z$, which implies that

$$(J_+)^\dagger = (J_x + iJ_y)^\dagger = J_x - iJ_y = J_-.$$

Multiplying $J_+|m\rangle = c_{m+1}|m+1\rangle$ from the left by $\langle m+1|$ and normalizing the states by $\langle m|m\rangle = 1$, we obtain

$$\langle m+1|J_+|m\rangle = c_{m+1}.$$

Complex conjugating this gives us $c^*_{m+1} = \langle m|J_-|m+1\rangle = b_m$, that is, $b_{m-1} = c^*_m$, so that we can write

$$J_-|m\rangle = c^*_m|m-1\rangle.$$

Acting on this with $J_+$ gives

$$J_+J_-|m\rangle = c^*_m J_+|m-1\rangle = |c_m|^2|m\rangle.$$

Similarly, acting with $J_-J_+$ on $|m\rangle$ gives

$$J_-J_+|m\rangle = c_{m+1}|m+1\rangle \implies |c_{m+1}|^2|m\rangle.$$

Since we know that the representation is finite dimensional, the ladder must terminate, that is, there must be a top rung. So, call the maximum value of $m$ by $j$. At this stage, all we know is that $j$ is a real number. (Note that we have not assumed that $m$ is an integer.) Thus, there is a state $|j\rangle$ such that $J_+|j\rangle = 0$. It corresponds to the top rung of the ladder.

At this point, we have only used the first part of (10.28). Now we use the second half:

$$\langle j|J_-J_+|j\rangle = \langle j|J_+J_- - 2J_z|j\rangle = |c_j|^2 - 2j,$$

thus determining $|c_j|^2 = 2j$. Furthermore,

$$2m = \langle m|2J_z|m\rangle = \langle m|[J_+, J_-]|m\rangle = \langle m|(J_+J_- - J_-J_+)|m\rangle = |c_m|^2 - |c_{m+1}|^2$$

We obtain a recursion relation

$$|c_m|^2 = |c_{m+1}|^2 + 2m,$$

which, together with $|c_j|^2 = 2j$, allows us to determine the unknown $|c_m|$. Here we go:

$$|c_{j-1}|^2 = |c_j|^2 + 2(j-1) = 2(2j-1),$$

then

$$|c_{j-2}|^2 = |c_{j-1}|^2 + 2(j-2) = 2(3j-1-2),$$

and eventually

$$|c_{j-s}|^2 = 2j + \sum_{i=1}^{s} 2(j-i) = 2((s+1)j - \sum_{i=1}^{s} i).$$

Recall the Gauss formula $\sum_{i=1}^{s} i = \frac{1}{2}s(s+1)$, and obtain

$$|c_{j-s}|^2 = 2((s+1)j - \frac{1}{2}s(s+1)) = (s+1)(2j-s).$$

We keep climbing down the ladder, increasing $s$ by 1 at each step. When $s = 2j$, we see that $c_{-j}$ vanishes. We have reached the bottom of the ladder. More explicitly, we have

$$J_-|-j\rangle = c^*_{-j}|-j-1\rangle = 0,$$

according to what we just derived. The minimum value of $m$ is $-j$. Since $s$ counts the number of rungs climbed down, it is necessarily an integer, and thus the condition $s = 2j$ that the ladder terminates implies that $j$ is either an integer or a half-integer, depending on whether $s$ is even or odd. If the ladder terminates, then we have the set of states $|-j\rangle, |-j+1\rangle, \ldots, |j-1\rangle, |j\rangle$, which totals $2j+1$ states.

For example, for $j = 2$, these states are $|-2\rangle, |-1\rangle, |0\rangle, |1\rangle, |2\rangle$. Starting from $|2\rangle$, we apply $J_-$ four times to reach $|-2\rangle$. (We will do this explicitly later in this chapter.) To emphasize the dependence on $j$, we sometimes write the kets $|m\rangle$ as $|j, m\rangle$. Notice that the ladder is symmetric under $|m\rangle \to |-m\rangle$, a symmetry that can be traced to the invariance of the algebra in (10.20) under $J_x \to J_x, J_y \to -J_y$, and $J_z \to -J_z$ (namely, a rotation through $\pi$ around the $x$-axis).

**Mysterious Appearance of the Half-Integers.** But what about the representations of the algebra corresponding to $j$ = a half-integer? For example, for $j = \frac{1}{2}$, we have a $2 \cdot \frac{1}{2} + 1 = 2$-dimensional representation consisting of the states $\left|-\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2}\right\rangle$. We climb down from $\left|\frac{1}{2}\right\rangle$ to $\left|-\frac{1}{2}\right\rangle$ in one step. Certainly no sight of a 2-dimensional representation in chapter I.3! The mystery of the $j = \frac{1}{2}$ representation will be resolved in chapter IV.5 when we discuss $SU(2)$, but let's not be coy about it and keep the reader in suspense. I trust that most readers have heard that it describes the electron spin. We did not go looking for the peculiar number, it came looking for us.

It should not escape your notice that as a by-product of requiring the ladder to terminate, we have also determined $|c_m|^2$. Indeed, setting $s = j - m$, we had $|c_m|^2 = (j+m)(j-m+1)$. Recalling the definition of $c_m$, we obtain

$$J_+|m\rangle = c_{m+1}|m+1\rangle = \sqrt{(j+1+m)(j-m)}|m+1\rangle. \tag{10.34}$$

and

$$J_-|m\rangle = c_m^*|m-1\rangle = \sqrt{(j+1-m)(j+m)}|m-1\rangle. \tag{10.35}$$

As a mild check on the arithmetic, indeed $J_+|j\rangle = 0$ and $J_-|-j\rangle = 0$. You might also have noticed that, quite rightly, the phase of $c_m$ is not determined, since it is completely up to us to choose the relative phase of the kets $|m\rangle$ and $|m-1\rangle$. Beware that different authors choose differently. I simply take $c_m$ to be real and positive. Tables of the $c_m$s for various $j$s are available, but it's easy enough to write them down when needed. Note also that the square roots in (10.34) and (10.35) are related by $m \leftrightarrow -m$.

**Example of ladder operators.** For convenience, let's list here the two most common cases needed in physics. For $j = \frac{1}{2}$:

$$J_+\left|-\frac{1}{2}\right\rangle = \left|\frac{1}{2}\right\rangle, \quad J_-\left|\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle. \tag{10.36}$$

For $j = 1$:

$$J_+|-1\rangle = \sqrt{2}|0\rangle, \quad J_+|0\rangle = \sqrt{2}|1\rangle, \quad J_-|1\rangle = \sqrt{2}|0\rangle, \quad J_-|0\rangle = \sqrt{2}|-1\rangle. \tag{10.37}$$

Note that the (nonzero) $c_m$ for these two cases are particularly easy to remember (that is, if for some odd reason you want to): they are all 1 in one case, and $\sqrt{2}$ in the other. Let us also write down the $j = 2$ case for later use:

$$\begin{gathered} J_+|-2\rangle = 2|-1\rangle, \quad J_+|-1\rangle = \sqrt{6}|0\rangle, \quad J_+|0\rangle = \sqrt{6}|1\rangle, \quad J_+|1\rangle = 2|2\rangle, \\ J_-|2\rangle = 2|1\rangle, \quad J_-|1\rangle = \sqrt{6}|0\rangle, \quad J_-|0\rangle = \sqrt{6}|-1\rangle, \quad J_-|-1\rangle = 2|-2\rangle. \end{gathered} \tag{10.38}$$

*So you did all of this before in QP1 and might be wondering what is new so what have you learnt from this? We'll we've implicitly figured out how to write $J_+$ and $J_-$, and thereby also $J_+$ and $J_-$*

*in a $2j+1$ dimensional basis working only from the known commutation relationships between $J_x$, $J_y$ and $J_z$. Or, in group theoretic language, from the structure constants that define the Lie Algebra of 3D rotations, $SO(3)$, we have computed a $2j+1$ dimensional representation of the $SO(3)$ Lie algebra.*

## 10.3 Addition of Angular Momentum (e.g. reducing tensor products of so(3) representations into its irreps)

If you remember back to when we first looked at irreps of groups I walked you through the example of breaking a tensor product reducible representation into its irreducible representations. In particular, we considered the tensor product representation

$$SU(2) \otimes SU(2),$$

and showed that it is reducible since it admits two invariant subspaces: a one-dimensional antisymmetric subspace, spanned by the singlet state

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

and a three-dimensional symmetric subspace, spanned by the remaining triplet states. At the time, I briefly mentioned that this decomposition is intimately related to the addition of angular momentum in quantum mechanics. We will now return to this point.

Concretely, we are going study how to break down a tensor product representation of multiple representation of so(3) into a direct sum of irreducible representations.

**Goal.** Given two irreducible representations of so(3) with spins $j_1$ and $j_2$, we aim to decompose the tensor product

$$j_1 \otimes j_2$$

into a direct sum of irreducible representations and to construct explicitly the change of basis between the uncoupled basis

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

and the coupled basis

$$|J, M\rangle.$$

**Physically** this corresponds to changing from a description in which each subsystem has its own angular momentum to one in which the combined system is characterised by a single total angular momentum. Concretely, it replaces the uncoupled basis

$$|j, m\rangle \otimes |j', m'\rangle,$$

which describes the individual angular momenta of the subsystems, by the coupled basis

$$|J, M\rangle,$$

which diagonalises the total angular momentum operators of the full system. In this basis, the tensor product representation becomes a direct sum of irreducible representations, and the generators of so(3) act block-diagonally.

**Mathematically**, the decomposition of the tensor product $j \otimes j'$ into irreducible representations is known as the *Clebsch–Gordan decomposition.* It takes the form

$$|J, M\rangle = \sum_{m=-j}^{j} \sum_{m'=-j'}^{j'} \big(|j, m\rangle \otimes |j', m'\rangle\big) \langle j, m; j', m' \,|\, J, M\rangle, \tag{10.39}$$

where we introduce the shorthand

$$|j, m; j', m'\rangle \equiv |j, m\rangle \otimes |j', m'\rangle.$$

Here the numbers $\langle j, m; j', m' \,|\, J, M\rangle$ are the Clebsch–Gordan coefficients. They quantify how the coupled state $|J, M\rangle$ is expressed as a linear combination of product states. Since these coefficients vanish unless $m + m' = M$, the double sum reduces effectively to a single sum.

**Total generators.** On the tensor product space, the total angular momentum operators are defined by

$$J_k = J_k^{(1)} \otimes I + I \otimes J_k^{(2)}, \qquad k = x, y, z,$$

so that

$$J_z\big(|j_1, m_1\rangle \otimes |j_2, m_2\rangle\big) = (m_1 + m_2)\, |j_1, m_1\rangle \otimes |j_2, m_2\rangle,$$

and hence

$$M = m_1 + m_2.$$

**Allowed values of $J$.** The total angular momentum takes values

$$J \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2\}.$$

Accordingly, the tensor product decomposes as

$$j_1 \otimes j_2 = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} J,$$

exhibiting the splitting of the representation into invariant irreducible subspaces corresponding to fixed total spin $J$.

**Algorithm.**

1. Start with the maximal value $J = j_1 + j_2$. The highest weight state is uniquely fixed by

$$|J = j_1 + j_2,\, M = j_1 + j_2\rangle = |j_1, j_1\rangle \otimes |j_2, j_2\rangle.$$

2. Generate all remaining states in the same multiplet by repeated application of the total lowering operator

$$J_- = J_-^{(1)} \otimes I + I \otimes J_-^{(2)},$$

using

$$|J, M - 1\rangle \propto J_- |J, M\rangle,$$

followed by normalisation. This produces all states

$$|J, M\rangle, \qquad M = J, J - 1, \dots, -J.$$

3. Decrease the value of $J$ by one and repeat the construction. For each new $J$, the states are fixed by requiring orthogonality to all previously constructed states with the same value of $M$.

4. Continue iteratively until $J = |j_1 - j_2|$.

**Example (A):** $j = \frac{1}{2}, j' = \frac{1}{2}$

Lets start with the example of $j = 1/2$ and $j' = -1/2$. We will explicitly derive the decomposition

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0.$$

(The notation $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ means that the tensor product of two spin-$\frac{1}{2}$ representations of so(3) decomposes into a direct sum of two irreducible representations: a three-dimensional spin-1 (triplet) representation and a one-dimensional spin-0 (singlet) representation.)

We begin by constructing the states with total angular momentum $J = 1$. The highest-weight state must satisfy $M = 1$, and since $m + m' = 1$, the only possibility is

$$|1,1\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle.$$

To obtain the $|1,0\rangle$ state, we apply the total lowering operator

$$J_- = J_-^{(1)} \otimes I + I \otimes J_-^{(2)}.$$

Using the standard relation $J_-|J,M\rangle = \sqrt{(J+M)(J-M+1)}\,|J,M-1\rangle$, we have

$$|1,0\rangle = \frac{1}{\sqrt{2}} J_-|1,1\rangle.$$

Therefore,

$$\begin{aligned} |1,0\rangle &= \frac{1}{\sqrt{2}}\Big(J_-^{(1)} \otimes I + I \otimes J_-^{(2)}\Big)\left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle \\ &= \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\right). \end{aligned}$$

Applying $J_-$ once more gives the final state in the triplet:

$$|1,-1\rangle = \left|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle.$$

We have thus obtained the three symmetric $J = 1$ states:

$$\begin{aligned} |1,1\rangle &= \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle, \\ |1,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\right), \\ |1,-1\rangle &= \left|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle. \end{aligned}$$

The remaining state must be orthogonal to $|1,0\rangle$ and therefore corresponds to $J = 0, M = 0$. It is given by the antisymmetric combination

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\right).$$

Hence the tensor product space decomposes as the direct sum of a triplet ($J = 1$) and a singlet ($J = 0$), proving

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0.$$

**Physics of angular momentum interpretation.** Physically, this result shows that two spin-$\frac{1}{2}$ particles can combine only into either a triplet state with total spin $J = 1$ or a singlet state with total spin $J = 0$. The triplet states correspond to symmetric combinations of the individual spins and describe aligned or partially aligned spins, while the singlet state is an antisymmetric combination describing a pair of spins coupled to zero total angular momentum.

**Example (B):** $j = 1$ **and** $j' = 1/2$

This time we take $j_1 = 1$ and $j_2 = \frac{1}{2}$. As before, we write the uncoupled basis as

$$|m_1\rangle \otimes |m_2\rangle \equiv \left|1, m_1; \tfrac{1}{2}, m_2\right\rangle,$$

where $m_1 \in \{1, 0, -1\}$ and $m_2 \in \{\frac{1}{2}, -\frac{1}{2}\}$.

**Step 1: States with** $J = \frac{3}{2}$**.** The highest-weight state has $M = \frac{3}{2}$, so the only possibility is

$$\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle = \left|1, 1; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle = |1\rangle \otimes \left|\tfrac{1}{2}\right\rangle.$$

To find $\left|\frac{3}{2}, \frac{1}{2}\right\rangle$, we apply the total lowering operator

$$J_- = J_-^{(1)} \otimes I + I \otimes J_-^{(2)}.$$

Using

$$J_-^{(1)}|1, 1\rangle = \sqrt{2}\,|1, 0\rangle, \qquad J_-^{(2)}\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle,$$

we obtain

$$J_-\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle = \sqrt{2}\left|1, 0; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \left|1, 1; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle.$$

We know $J_-\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2}, \frac{1}{2}\right\rangle$, so

$$\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}\left|1, 0; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \left|1, 1; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\right) = \sqrt{\tfrac{2}{3}}\left|1, 0; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1, 1; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle.$$

Applying $J_-$ again gives $\left|\frac{3}{2}, -\frac{1}{2}\right\rangle$. Using

$$J_-^{(1)}|1, 0\rangle = \sqrt{2}\,|1, -1\rangle, \qquad J_-^{(2)}\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = 0,$$

a short calculation yields

$$\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|1, 0; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1, -1; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle.$$

Finally, the lowest-weight state is uniquely determined as

$$\left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = \left|1, -1; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle.$$

**Step 2: States with $J = \frac{1}{2}$. Case $M = \frac{1}{2}$.** The condition $m_1 + m_2 = \frac{1}{2}$ leaves the two basis vectors

$$\left\{ \left|1,1;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle,\ \left|1,0;\tfrac{1}{2},\tfrac{1}{2}\right\rangle \right\}.$$

From the previous step we already found

$$\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|1,0;\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1,1;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle.$$

We now look for a normalized linear combination of the same basis vectors that is orthogonal to this state. Let

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = a\left|1,0;\tfrac{1}{2},\tfrac{1}{2}\right\rangle + b\left|1,1;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle.$$

Orthogonality requires

$$\left\langle\tfrac{3}{2},\tfrac{1}{2}\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = 0,$$

which gives the condition

$$\sqrt{\tfrac{2}{3}}\,a + \sqrt{\tfrac{1}{3}}\,b = 0.$$

Choosing $a = \sqrt{\frac{1}{3}}$ and $b = -\sqrt{\frac{2}{3}}$ also ensures normalisation, and therefore

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|1,0;\tfrac{1}{2},\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1,1;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle.$$

**Case $M = -\frac{1}{2}$.** Now the condition $m_1 + m_2 = -\frac{1}{2}$ yields the basis

$$\left\{ \left|1,-1;\tfrac{1}{2},\tfrac{1}{2}\right\rangle,\ \left|1,0;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \right\}.$$

From Step 1 we already have

$$\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|1,0;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1,-1;\tfrac{1}{2},\tfrac{1}{2}\right\rangle.$$

As before, we seek a normalized vector orthogonal to this one. Writing

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = a\left|1,0;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + b\left|1,-1;\tfrac{1}{2},\tfrac{1}{2}\right\rangle,$$

the orthogonality condition becomes

$$\sqrt{\tfrac{2}{3}}\,a + \sqrt{\tfrac{1}{3}}\,b = 0.$$

Choosing again $a = \sqrt{\frac{1}{3}}$ and $b = -\sqrt{\frac{2}{3}}$, we obtain

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|1,0;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1,-1;\tfrac{1}{2},\tfrac{1}{2}\right\rangle.$$

We have therefore explicitly constructed the two states forming the spin-$\frac{1}{2}$ multiplet by taking, for each $M$, the normalized vector orthogonal to the corresponding spin-$\frac{3}{2}$ state in the same two-dimensional subspace.

Collecting everything, the tensor product space decomposes into the four-dimensional $J = \frac{3}{2}$ multiplet and the two-dimensional $J = \frac{1}{2}$ multiplet, thus explicitly realizing

$$1 \otimes \tfrac{1}{2} = \tfrac{3}{2} \oplus \tfrac{1}{2}.$$

Now it probably a good moment to look back at the algorithm for the general case and see if it makes more sense than when you read it the first time. You'll also gets lots more practice of this in the problem sheets.

## 10.4 Other applications of Group Theory and Lie Algebras

To end, I just want to highlight that Lie Groups and Lie Algebras appear all over the place. I've focused on their application in angular momentum because this should be most familiar given what you've seen before- and is important to understand in a lot of atomic and molecular physics. But let me just name drop a few other applications and give you a few references as to where you can read more about them. (This is, of course, all non-examinable.)

- **Particle Physics.** The most obvious area where you really need to understand Lie Groups and Lie Algebras in Particle Physics. In fact, if you decide to focus on this in your master's you will start with a TPIV devoted to learning Lie Algebra for Particle physics via this textbook.

- **Controlling quantum systems.** You've seen that Lie Algebras are all about figuring out what Hermitian operators generate what unitary representations of a group... this means you can use your understanding of Lie Algebras to figure out how to design Hamiltonians to implement various unitaries on that systems, that is, how to control that system. This is important if you are an theorist/experimentalist trying to build a quantum computer (or other quantum technology). It is also important if you are a quantum software developer trying to design certain quantum algorithms. For an introduction see this tutorial.

- **Machine Learning (Quantum and Classical)** Why is symmetry important in machine learning? This is explained very nicely in this blog post. Consider everyone's favourite example of a machine learning task: classifying images to decide if they include cats of dogs. (If you want a less inane task consider trying to classify whether an images of tumours contain cancerous cells. Or whether images of galaxies contain supernova.)

  There are many different transformations one can perform to an image of a cat that still leave it as a picture of a cat - e.g. you can rotate it or reflect it and you are still left with an image of a cat (Fig. 10.1).

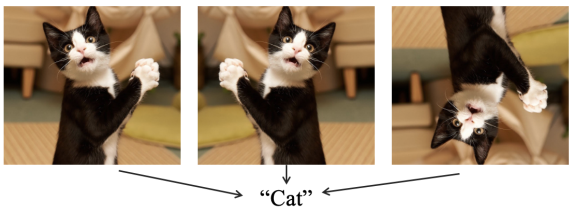


Figure 10.1: A picture of a rotated cat or flipped cat is still a picture of a cat.

  We want our classifier to be *invariant* under these symmetry transformations. In the context of image processing (or modelling molecules or materials) these symmetry transformations will typically be geometric transformations. Beyond image classification other symmetry transformations, such as permutation invariance, can become important. And, of course, mathematically all these symmetry transformations can represented by the actions of elements of a symmetry group. The theory of Lie Algebras (and group/rep theory more generally) provides us with a way of constructing models with these symmetries in built. For more information on this take a look at my notes from last year, check out this (quite technical) tutorial or this (less technical) tutorial.

- **Classically simulating quantum systems.** As we've discussed before, simulating quantum systems classically is generally hard because it involves multiplying together exponentially large matrices. But if your system as symmetries you can use clever tricks from the theory of Lie Algebras and Lie Groups to make this easier. See this tutorial for more information.

Let $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \in \mathcal{H}_1$, $|\phi\rangle \in \mathcal{H}_2$, $\dim(\mathcal{H}_1) > \dim(\mathcal{H}_2)$

| | |
|---|---|
| $\|\psi_1\rangle$ | $\|\psi_1\rangle \|\phi\rangle$ |
| $\|\psi_1\rangle \|\psi_2\rangle$ | $\|\psi_1\rangle \langle\psi_3\|$ |

Table 1: Is this loss?

Figure 10.2: Let's have another meme. I originally gave this the wooden spoon award because my reaction, similarly to many of you I guess, was 'is this even a meme?'. But having now had it explained to me I have to concede its pretty clever. And if you don't get it - that's just a healthy sign that you don't spend too too much time online.